EXTERNAL DEFINABILITY AND GROUPS IN NIP THEORIES

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Abstract. We prove that many properties and invariants of definable groups in NIP theories, such as definable amenability, G/G^{00}, etc., are preserved when passing to the theory of the Shelah expansion by externally definable sets, M^{ext}, of a model M. In the light of these results we continue the study of the "definable topological dynamics" of groups in NIP theories. In particular we prove the Ellis group conjecture relating the Ellis group to G/G^{00} in some new cases, including definably amenable groups in o-minimal structures.

1. Introduction

The class of NIP theories (theories without the independence property) is a common generalization of stable and o-minimal theories (also includes algebraically closed valued fields and p-adics). One of the equivalent definitions requires that every family of uniformly definable sets has finite Vapnik-Chervonenkis dimension. Groups definable in NIP theories were studied in [HPP08, HP11] for example, and this study generalizes both the theory of stable [Poi01] and o-minimal groups. Recently notions of topological dynamics were brought into the picture by Newelski, for example [New09], and later by Pillay, for example [Pil13]. In [GPP12b], Gismatullin, Penazzi and Pillay developed a basic theory built around the notion of a *definable* action of a group G definable in a model M, on a compact space X, but under a (weak) definability of types assumption on M. In the case of stable theories all types over all models are definable, so the assumption is automatically satisfied. We want to be able to apply the topological dynamics notions (discussed in section 4) to groups definable in NIP theories. While of course in models of an NIP theory externally definable sets need not be internally definable, their behavior turns out to be somewhat tame and approximable by internally definable sets. A fundamental result in this direction is Shelah's theorem on externally definable sets:

Fact 1.1. [She09] *Let M be a model of an* NIP *theory T.*

(1) *The projection of an externally definable subset of M is externally definable.*

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(2) *In particular* $\mathrm{Th}\left(M^{\mathrm{ext}}\right)$ *eliminates quantifiers, and is* NIP.

Further study of externally definable sets in NIP theories, as well as a refined and uniform version of Shelah's theorem, can be found in [CS12, CS].

So one aim of this paper is to show that many properties of (e.g. definable amenability) and objects attached to (e.g. G^{00}) a group G definable over a model M of an NIP theory T are preserved when passing to $\mathrm{Th}\left(M^{\mathrm{ext}}\right)$, answering some questions raised in [GPP12b]. A second aim of this paper, bearing in mind the above, is to prove some more cases of the "Ellis group" conjecture (originating with Newelski) which says that in the NIP environment, for suitable groups G definable over a model M, G/G^{00} should coincide with the "Ellis group" computed in $\mathrm{Th}\left(M^{\mathrm{ext}}\right)$, where all types over M^{ext} are definable. And as is shown in the first part of the paper G^{00} is unchanged when passing to the expanded theory. So the problem is well-defined, and we answer it in particular for definably amenable groups in *o*-minimal theories, as well as dp-minimal groups. We also study "topological dynamical" properties of groups with "definable f-generics" (see below), complementing the study in [Pil13] of groups with finitely satisfiable generics.

Now for some more details.

In Section 2 we establish a couple of general facts about measures in NIP theories. We show in Theorem 2.5 that every measure over a small model in an NIP theory has a global invariant extension which is also an heir (generalizing the result for types from [CK12]). We also observe that the answer to [GPP12b, Question 3.15] is positive in the case of NIP theories.

Theorem (2.7). (1) *Assume that* T *is* NIP, $M \models T$ *and all types over* M *are definable. Then every Borel probability measure on* $S(M)$ *is definable (a measure* μ *is definable if for every* L*-formula* $\phi(x,y)$ *and closed disjoint subsets* C_1, C_2 *of* $[0,1]$*, the sets* $\{b \in M : \mu(\phi(x,b)) \in C_1\}$ *and* $\{b \in M : \mu(\phi(x,b)) \in C_2\}$ *are separated by a definable set in* M*).*

(2) *In particular, if* G *is a definably amenable* M*-definable group, then it is witnessed by an* M*-definable measure.*

Examples of structures satisfying the assumption of the theorem are: any model of a stable theory, $(\mathbb{R}, +, \cdot)$, $(\mathbb{Q}_p, +.\cdot)$, $(\mathbb{Z}, +, <)$ (see [CS, Section 5] for a discussion of this phenomenon).

In Section 3 we study lifting of Keisler measures and related objects to Shelah's expansion. The main theorem is:

Theorem (3.17). *Assume that* T *is* NIP, $M \models T$ *and* G *is an* M*-definable group.*

(1) *Assume that* G *is definably amenable, i.e., there is a Borel probability measure* μ *on* $S_G(M)$ *invariant under the action of* $G(M)$*. Then* G *is still*

definably amenable in the sense of M^{ext}: there is some Borel probability measure μ' on $S_G(M^{\text{ext}})$ extending μ and $G(M)$-invariant.

(2) *Assume that G is definably extremely amenable, i.e., the action of $G(M)$ on $S_G(M)$ has a fixed point p. Then G is still definably extremely amenable in the sense of M^{ext}: there is some $p' \in S_G(M^{\text{ext}})$ extending p and $G(M)$-invariant.*

This answers positively [GPP12b, Question 3.16, (1) and (2)]. We remark that (1) was essentially known for o-minimal theories but (2) was open even in the o-minimal case. Our proof combines the existence of invariant heirs for measures from Section 2 (as explained in Section 3.1) along with the existence of a canonical continuous retraction from the space of global invariant measures onto the closed subspace of finitely satisfiable measures (Sections 3.2, 3.3).

If G is a group definable over model M of an NIP theory, then definable amenability of G is equivalent to the existence of a global f-generic type of G, namely a complete type p over the monster model $\mathbb{M}$, every left translate gp of which does not fork over M (equivalently is $Aut(\mathbb{M}/M)$-invariant). An f-generic type p can fall into one of the two extreme cases: (a) p is fsg (with respect to M), namely every left translate gp is finitely satisfiable in M, and (b) p is a definable (over M) f-generic, namely p is f-generic with respect to M and is definable over M, *equivalently* every gp is definable over M. So we will observe that both these extreme witnesses of definable amenability are preserved when passing to $\mathrm{Th}(M^{\text{ext}})$.

Theorem (3.19)**.** *Suppose T is* NIP*, $M \models T$ and G is a group definable over M.*

(1) *If G has a global* fsg *type (with respect to M), then G has a global* fsg *type with respect to M^{ext} in* $\mathrm{Th}(M^{\text{ext}})$*.*

(2) *If G has a global f-generic which is definable over M, then the same is true for* $\mathrm{Th}(M^{\text{ext}})$*.*

We also characterize definably extremely amenable groups as those definably amenable groups in which $G^{00} = G$.

In Section 4 of the paper we study the effect of externally definable sets on the model-theoretic connected components of definable groups. Working in a monster model, let G be a definable group and $A \subseteq \mathbb{M}$. Recall that G^0_A, G^{00}_A and G^{∞}_A are defined respectively as the intersection of all subgroups of G of finite index definable over A, the intersection of all subgroups of G of bounded index type-definable over A, and the intersection of all subgroups of G of bounded index invariant over A. The subscript is omitted if $A = \emptyset$. A fundamental fact is:

Fact 1.2. *Let T be* NIP *and let G be a definable group.*

(1) [She08] $G_A^{00} = G_\emptyset^{00}$ *for every small set* A*, so in particular the intersection of all type-definable subgroups of bounded index is type definable over* $\emptyset$ *and is of index* $\leq 2^{|T|}$.
(2) [She07] *for the abelian case, and* [Gis11] *in general:* $G_A^\infty = G_\emptyset^\infty$ *for every small set* A*, so in particular the intersection of all subgroups of bounded index invariant over small subsets is invariant over* $\emptyset$ *and is of index* $\leq 2^{|T|}$.

Of course $G^0 \subseteq G^{00} \subseteq G^\infty$, and there are NIP examples where $G^{00} \subsetneq G^\infty$ [CP12].

So let M be a model of an NIP theory and let $G(M)$ be an M-definable group; to simplify the notation we assume that G is the whole universe. Assume that H is an externally definable subgroup of M (i.e., $H = M \cap \phi(x)$ for some $\phi(x) \in L(\mathbb{M})$ is a subgroup of M). In general it need not contain any M-definable subgroups:

Example 1.3. Let $M \succ (\mathbb{R}, +, \cdot)$ be $\left(2^{\aleph_0}\right)^+$-saturated. Then M contains the subgroup $H = \left\{x \in M : \bigwedge_{r \in \mathbb{R}} |x| < r\right\}$ of infinitesimal elements. Note that H is externally definable as "$M \cap (c < x < d)$" where $c, d \in \mathbb{M}$ realize the appropriate cuts of M. However H does not contain any M-definable subgroups.

However we show that these connected components are not affected by adding externally definable sets:

Theorem. *Let* T *be an* NIP *theory in the language* L*, and* $M \models T$*. Let* $T' = \mathrm{Th}(M^{\mathrm{ext}})$*, and let* $\mathbb{M}'$ *be a monster model of* T'*. Let* $\mathbb{M} = M' \restriction L$ *— a monster model of* T*. Note that* T' *is* NIP *by Fact 1.1. Then we have:*
(1) $G^0(\mathbb{M}) = G^0\left(\mathbb{M}'\right)$ *(Theorem 4.5),*
(2) $G^{00}(\mathbb{M}) = G^{00}\left(\mathbb{M}'\right)$ *(Corollary 4.16),*
(3) $G^\infty(\mathbb{M}) = G^\infty\left(\mathbb{M}'\right)$ *(Corollary 4.21).*

For the proof we first establish existence of the corresponding connected components relatively to a predicate and a sublanguage, and then we show that each of these relative connected components coincides with the corresponding connected component of the theory induced on the predicate.

Corollary (4.6)**.** *Let* T *be* NIP *and let* M *be a model. Assume that* G *is an externally definable subgroup of* M *of finite index. Then it is internally definable.*

Finally in Section 5 of the paper we return to the motivating context of "tame topological dynamics". We will explain the set-up in some detail in Section 5. But we give here a brief description of the notions so as to be able to state the main results. The context here is an NIP theory T, model M of T and definable group G, defined over M. The type space $S_G(M^{\mathrm{ext}})$ is acted on "definably" by $G(M) = G(M^{\mathrm{ext}})$, and also has a canonical semigroup structure, continuous in the first coordinate. There exist minimal closed G-invariant subsets of the type space, and elements of such "minimal subflows" are called *almost periodic types*. Any two minimal closed $G(M)$-invariant subspaces of $S_G(M^{\mathrm{ext}})$ are "isomorphic" and coincide with the unique universal minimal definable $G(M)$-flow. So this is a rather basic invariant of G (or rather $G(M)$) given by the set-up of topological dynamics. In [Pil13] it was proved that when G has fsg then there is a unique minimal closed $G(M)$-invariant subspace of $S_G(M^{\mathrm{ext}})$ and it coincides with the set of generic types. We will study the other extreme case of definable amenability, when G has an f-generic type, definable over M. And we prove as a part of Proposition 5.6:

Theorem. *Suppose G has a global f-generic, definable over M. Then for $p \in S_G(M^{\mathrm{ext}})$, p is almost periodic if and only if p is a "definable f-generic" in the sense that the unique global heir of p is f-generic.*

As proved in the previous section, G/G^{00} is the same whether computed in T or in $Th(M^{\mathrm{ext}})$ and we just write G/G^{00}. Fix a minimal closed $G(M)$-invariant subspace $\mathcal{M}$ of $S_G(M^{\mathrm{ext}})$, and an idempotent $u \in \mathcal{M}$. Then $u\mathcal{M}$ is a subgroup of $S_G(M^{\mathrm{ext}})$, which we call the Ellis group (attached to M, G) and whose isomorphism type does not depend on the choice of $\mathcal{M}$ or u. In fact there is also a certain non-Hausdorff topology on $u\mathcal{M}$, but it will not concern us in the current paper. The canonical surjective homomorphism $G \to G/G^{00}$ factors naturally through the space $S_G(M^{\mathrm{ext}})$, namely we have a well-defined continuous surjection $\pi : S_G(M^{\mathrm{ext}}) \to G/G^{00}$ taking $\mathrm{tp}(g/M)$ to the coset gG^{00}, and the restriction of π to the group $u\mathcal{M}$ is a surjective homomorphism. We will say that the Ellis group $u\mathcal{M}$ coincides with (or equals) G/G^{00} if $\pi|u\mathcal{M}$ is an isomorphism. It was suggested by Newelski in [New09] that in many cases, $u\mathcal{M}$ does equal G/G^{00}. Let us formalize this by conjecturing that when G is definable amenable, then $u\mathcal{M}$ equals G/G^{00}. This was essentially proved in [Pil13] when G is fsg. We will prove some more cases in Sections 5.4 – 5.8:

Theorem 1.4. *Under the assumptions above, the Ellis group coincides with G/G^{00} in the following cases:*

1. *G is definably extremely amenable.*
2. *G is* fsg.
3. *G has a definable f-generic with respect to M.*
4. *G is definably amenable, dp-minimal.*
5. *T is an o-minimal expansion of a real closed field and G is definably amenable (computed in T or equivalently in $Th(M^{\mathrm{ext}})$).*

Notation. We will write $N \succ^{+} M$ to denote that N is an elementary extension of M which is moreover $|M|^{+}$-saturated. As usual, $S(A)$ denotes the space of types over A. We also write $S^{\mathrm{inv}}(A, B)$ (resp. $S^{\mathrm{fs}}(A, B)$) for the set of types over A

invariant (resp. finitely satisfiable) over B. Both are closed subsets of $S(A)$ as $S^{\mathrm{fs}}(A,B)$ is the closure of the set of types over A realized in B and

$$S^{\mathrm{inv}}(A,B)=\left\{p\in S(A):\bigwedge_{a\equiv_B a',\phi(x,y)\in L(B)}(p\vdash\phi(x,a)\leftrightarrow\phi(x,a'))\right\}.$$

Whenever we say measure, we mean a finitely additive Keisler measure (equivalently, a regular Borel probability measure on $S(A)$), see e.g. [HP11]. For a measure μ we denote by $S(\mu)$ its support: the set of types weakly random for μ, i.e. the closed set of all p such that for any $\phi(x)$, $\phi(x)\in p$ implies $\mu(\phi(x))>0$. We will assume some basic knowledge of forking for types and measures. E.g., in NIP a type does not fork over a model if and only if it is invariant over it, etc [HP11, CK12].

2. Existence of invariant heirs and definability of measures

2.1. **Existence of global invariant heirs for measures over models.** We recall some facts about forking and dividing in NIP theories from [CK12].

Fact 2.1. *Let T be* NIP *and $M\models T$.*

1. *A formula $\phi(x,a)\in L(\mathbb{M})$ forks over M if and only if it divides over M.*
2. *Every type $p(x)\in S(M)$, where x can be a tuple of variables of arbitrary length, admits a global extension which is both M-invariant and an heir over M.*

The aim of this section is to generalize (2) to arbitrary measures in NIP theories, i.e. to demonstrate that every measure over a model of an NIP theory admits a global invariant heir.

Definition 2.2. We say that $\nu\in\mathfrak{M}(\mathbb{M})$ is an *heir* of $\mu\in\mathfrak{M}(M)$ if for any finitely many formulas $\phi_0(x,a),\dots,\phi_n(x,a)\in L(\mathbb{M})$ and $\varepsilon_0,\dots,\varepsilon_n\in[0,1)$, if $\bigwedge_{i\le n}(\nu(\phi_i(x,a))>\varepsilon_i)$ then $\bigwedge_{i\le n}(\mu(\phi_i(x,b))>\varepsilon_i)$ for some $b\in M$.

Remark 2.3.
1. Note that for types we recover the usual notion of an heir.
2. A weaker notion of an heir of a measure was defined in [HPP08, Remark 2.7], but working with that definition does not seem sufficient for our purposes.

Given $p_0,\dots,p_{n-1}\in S(A)$ and $\phi(x,a)\in L(A)$, we set $\mathrm{Av}(p_0,\dots,p_{n-1};\phi(x,a))=\frac{|\{i<n:\phi(x,a)\in p_i\}|}{n}$. Then one defines $\mu=\mathrm{Av}(p_0,\dots,p_{n-1})\in\mathfrak{M}(A)$, the average measure of $p_0,\dots,p_{n-1}$, by setting $\mu(\phi(x,a))=\mathrm{Av}(p_0,\dots,p_{n-1};\phi(x,a))$ for all $\phi(x,a)\in L(A)$.

The following is a corollary of the VC-theorem from [HP11, Section 4]. It is stated there for a single formula, but easily generalizes to a finite set of formulas by encoding them into one.

Fact 2.4. *Let T be* NIP*, μ a measure on $S(A)$, $\Delta(x) = \{\phi_i(x, y_i)\}_{i<m}$ a finite set of L-formulas, and let $\varepsilon > 0$ be arbitrary. Then there are some types $p_0, \ldots, p_{n-1} \in S(A)$ such that for every $a \in A$ and $\phi(x, y) \in \Delta$, we have*

$$|\mu(\phi(x, a)) - \mathrm{Av}(p_0, \ldots, p_{n-1}; \phi(x, a))| \leq \varepsilon.$$

Furthermore, we may assume that $p_i \in S(\mu)$, the support of μ, for all $i < n$.

Theorem 2.5. *Let T be* NIP *and $M \models T$. Then every $\mu \in \mathfrak{M}(M)$ has a global extension $\nu \in \mathfrak{M}(\mathbb{M})$ which is both invariant over M and an heir of μ.*

Proof. Let $\mu \in \mathfrak{M}(M)$ be given, let Δ be a finite set of formulas, and we define the following subsets of $\mathfrak{M}(\mathbb{M})$:

$$I = \{\nu \in \mathfrak{M}(\mathbb{M}) : \nu(\phi(x, a)) = \nu(\phi(x, b)) : a \equiv_M b \in \mathbb{M}, \phi(x, y) \in L(M)\},$$

$$H_{\mu,\Delta,\varepsilon} = \left\{ \nu \in \mathfrak{M}(\mathbb{M}) : \bigvee_{\phi \in \Delta} \nu(\phi(x, a)) \leq r \text{ for all } r \in [0, 1), \phi(x, y) \in L(M) \right.$$
$$\left. \text{such that } \bigvee_{\phi \in \Delta} \mu(\phi(x, b)) \leq r - \varepsilon \text{ for all } b \in M \right\}.$$

So $H_{\mu,\Delta,\varepsilon}$ is the set of global Δ-heirs of μ up to an ε-mistake and I is the set of M-invariant global measures. Let also H_μ be the set of global heirs of μ. Note that all these sets are closed in $\mathfrak{M}(\mathbb{M})$, that $H_\mu = \bigcap_{\Delta \subseteq L \text{ finite}, n \in \omega} H_{\mu,\Delta,\frac{1}{n}}$ and that every element of H_μ extends μ.

Fix an arbitrary $\varepsilon > 0$ and finite $\Delta(x) \subseteq L$. By Fact 2.4 there are some $p_0, \ldots, p_{n-1} \in S(M)$ such that $|\mu(\phi(x, a)) - \mathrm{Av}(p_0, \ldots, p_{n-1}; \phi(x, a))| \leq \varepsilon$ for all $a \in M$ and $\phi(x, y) \in \Delta$. Let $p(x_0, \ldots, x_{n-1}) \in S_n(M)$ be some completion of $p_0(x_0) \cup \ldots \cup p_{n-1}(x_{n-1})$, then by Fact 2.1(2) there is some $q(x_0, \ldots, x_{n-1}) \in S_n(\mathbb{M})$ — a global M-invariant heir of $p(x_0, \ldots, x_{n-1})$. Let $q_i = q \restriction x_i \in S(\mathbb{M})$ for $i < n$, and let $\nu_{\varepsilon,\Delta} = \mathrm{Av}(q_0, \ldots, q_{n-1}) \in \mathfrak{M}(\mathbb{M})$.

Claim. *$\nu_{\varepsilon,\Delta} \in I$, i.e. it is invariant over M.*

Proof. By NIP enough to show that ν does not fork over M. If it does then $\nu(\phi(x, a)) > 0$ for some $\phi(x, a)$ forking over M, which by the definition of $\nu_{\varepsilon,\Delta}$ implies that $\phi(x, a) \in q_i$ for some $i < n$ — a contradiction. $\square$

Claim. *$\nu_{\varepsilon,\Delta} \in H_{\mu,\Delta,\varepsilon}$.*

Proof. Assume that $\bigwedge_{\phi \in \Delta} (\nu_{\varepsilon,\Delta}(\phi(x, a)) > r_\phi)$ holds for some $a \in \mathbb{M}$ and $r_\phi \in [0, 1)$. Then $\bigwedge_{\phi \in \Delta} \left(\frac{|\{i<n : \phi(x,a) \in q_i\}|}{n} > r_\phi \right)$, that is $\bigwedge_{\phi \in \Delta} \left(\frac{|\{i<n : \phi(x_i,a) \in q\}|}{n} > r_\phi \right)$. As q is an heir over M, there is some $b \in M$ satisfying $\bigwedge_{\phi \in \Delta} \left(\frac{|\{i<n : \phi(x_i,b) \in p\}|}{n} > r_\phi \right)$, so $\bigwedge_{\phi \in \Delta} \left(\frac{|\{i<n : \phi(x,b) \in p_i\}|}{n} > r_\phi \right)$. But then by the choice of p_i's it follows that $\mu(\phi(x, b)) > r_\phi - \varepsilon$ for every $\phi \in \Delta$, as wanted. $\square$

Finally, assume towards a contradiction that $H_\mu \cap I$ is empty. But then it follows by compactness of $\mathfrak{M}(\mathbb{M})$ that $H_{\mu,\Delta,\frac{1}{n}} \cap I$ is empty for some finite $\Delta \subseteq L$ and $n \in \omega$. However, $\nu_{\frac{1}{n},\Delta} \in H_{\mu,\Delta,\frac{1}{n}} \cap I$ by the previous claims. $\square$

Example 2.6. Every M-definable measure $\mu \in \mathfrak{M}(\mathbb{M})$ is an invariant heir over M.

2.2. **Definability of types implies definability of measures.** We give another application of Fact 2.4 and show that if all types over a model are definable, then measures over it are definable as well.

Theorem 2.7. (1) *Assume that T is* NIP, *$M \models T$ and all types over M are definable. Then every Borel probability measure on $S(M)$ is definable.*
(2) *In particular, if G is a definably amenable M-definable group, then it is witnessed by an M-definable measure.*

Proof. We are assuming that all types over M are definable, and let μ be a measure on $S(M)$. We want to show that μ is definable. Let $\phi(x,y)$ and C_1, C_2 closed disjoint subsets of $[0,1]$ be given. It then follows that there is some $\epsilon > 0$ such that no point of C_1 has any point of C_2 in its ϵ-neighbourhood. Let $D_i = \{b \in M : \mu(\phi(x,b)) \in C_i\}$ for $i \in \{0,1\}$.

Let $p_0, \dots, p_{n-1} \in S(M)$ by as given by Fact 2.4 for ϕ, μ and $\frac{\epsilon}{2}$. As each of p_i's is definable, there is some $d_\phi p_i(y) \in L(M)$ such that for any $a \in M$ we have $\phi(x,a) \in p_i \Leftrightarrow M \models d_\phi p_i(a)$. Note that $\mathrm{Av}(p_0, \dots, p_{n-1}; \phi(x,a))$ can only take values from the finite set $\left\{\frac{m}{n} : m < n\right\}$, and let C be the set of those values whose distance from C_1 is less that $\frac{\epsilon}{2}$. Let $D = \{a \in M : \mathrm{Av}(p_0, \dots, p_{n-1}; \phi(x,a)) \in C\}$, it is definable by some boolean combination of $d_\phi p_i(y)$'s, so $L(M)$-definable. It is then easy to see from the definition that $D_1 \subseteq D$ and that $D \cap D_2 = \emptyset$, as wanted. $\square$

3. Lifting measures to Shelah's expansion and preservation of amenability

3.1. **Definable amenability and f-generic types.**

Definition 3.1. A definable group G is *definably amenable* if there is a left-invariant finitely additive probability measure defined on the algebra of all definable subsets of G.

First we summarize some known facts about definably amenable groups in NIP theories which will be used freely later on in the text.

Definition 3.2. A global type $p \in S_G(\mathbb{M})$ is *left f-generic* over a small model M if $g \cdot p$ does not fork over M for all $g \in G$ (equivalently, $g \cdot p$ is invariant over M for all $g \in G$).

Fact 3.3.
(1) [HP11, 5.10,5.11] *G is definably amenable if and only if for some (equivalently, any) small model M, there is a global type $p \in S(\mathbb{M})$ which is left f-generic over M.*
(2) [HPP08, Section 5] *Definable amenability is a property of the theory: If $S_G(M)$ admits a G-invariant measure and $M \equiv N$, then $S_G(N)$ admits a G-invariant measure.*
(3) [HP11, 5.6(i)] *If $p \in S(\mathbb{M})$ is f-generic then* $\mathrm{Stab}(p) = G^{00} = G^{\infty}$, *where* $\mathrm{Stab}(p) = \{g \in G : g \cdot p = p\}$.

Next we consider extending G-invariant (M-invariant) measures to larger sets of parameters.

Proposition 3.4. *Assume that $\mu \in \mathfrak{M}(M)$ is $G(M)$-invariant and $\nu \in \mathfrak{M}(\mathbb{M})$ is an heir of μ (in the sense of Definition 2.2). Then ν is $G(\mathbb{M})$-invariant.*

Proof. Assume not, then $\nu(\phi(x,a)) \neq \nu\left(\phi\left(g^{-1} \cdot x, a\right)\right)$ for some $\phi(x,a) \in L(\mathbb{M})$ and $g \in G(\mathbb{M})$. That is, $\nu(\phi(x,a)) > \varepsilon \wedge \nu\left(\neg\phi\left(g^{-1} \cdot x, a\right) \wedge g \in G\right) > 1-\varepsilon$ for some ε. As ν is an heir of μ, this implies that $\mu(\phi(x,b)) > \varepsilon \wedge \mu\left(\neg\phi\left(h^{-1} \cdot x, b\right)\right) > 1-\varepsilon$ for some $b \in M$ and $h \in G(M)$, contradicting $G(M)$-invariance of μ. □

Proposition 3.5. *If T is* NIP, *$M \models T$ and $\mu \in \mathfrak{M}(M)$ is $G(M)$-invariant, then there is some $\nu \in \mathfrak{M}(\mathbb{M})$ extending μ, which is both $G(\mathbb{M})$-invariant and M-invariant.*

Proof. By Theorem 2.5, μ admits a global M-invariant heir ν. By Proposition 3.4 ν is G-invariant. □

Finally for this section, we characterize definable extreme amenability.

Definition 3.6. A definable group G is *definably extremely amenable* if there is a G-invariant type $p \in S_G(\mathbb{M})$.

It is easy to see that definable extreme amenability is a property of the theory: by compactness, if there is a $G(M)$-invariant type in $S_G(M)$ and $M \equiv N$, then there is a $G(N)$-invariant type in $S_G(N)$.

Proposition 3.7. *G is definably extremely amenable if and only if it is definably amenable and $G = G^{00}$.*

Proof. If G is definably amenable, then there is an f-generic p such that $\mathrm{Stab}(p) = G^{00} = G$. But then p is G-invariant, so G is definably extremely amenable.
Conversely, as G is definably extremely amenable, for some small model M there is some $p \in S(M)$ which is $G(M)$-invariant. Let $p^* \in S(\mathbb{M})$ be a non-forking heir of p. It follows that p^* is $G(\mathbb{M})$-invariant and M-invariant, so in particular f-generic over M. But then $G^{00} = \mathrm{Stab}(p^*) = G$. □

Remark 3.8. In particular, if T is stable then G is definably extremely amenable if and only if $G = G^0$. However, in the NIP case definable amenability does not follow even from $G = G^\infty$. Indeed, given a saturated real closed field K, $G = SL(2, K)$ is simple as an abstract group modulo its finite center. Then $G = G^\infty$, but this group is not definably amenable.

3.2. **Extracting the finitely satisfiable part of an invariant type.** We present a construction due to the third author from [Sim13].

Let $M \models T$ and $N \succ M$ be $|M|^+$-saturated. We let M^{ext} be a Shelah's expansion of M in the language $L' = \{R_\phi(x) : \phi(x) \in L(N)\}$ with $R_\phi(M) = M \cap \phi(x)$, $T' = \mathrm{Th}_{L'}(M^{\mathrm{ext}})$. Let $\left(N', M', (R_\phi)_{\phi \in L(N)}\right)$ be an $|N|^+$-saturated expansion of $\left(N, M, (R_\phi)_{\phi \in L(N)}\right)$ with a new predicate $\mathbf{P}(x)$ naming M. It follows that $M' \succ^{L'} M$ and that still $R_\phi(M') = M' \cap \phi(x)$. We can identify $M' \restriction L$ with the monster model of T.

Proposition 3.9. *Working in T', for every L-type $p \in S^{\mathrm{inv}}(M', M)$ and $R_\phi(x) \in L'$, if $p(x) \cup R_\phi(x)$ is consistent then $p(x) \vdash R_\phi(x)$ (and in fact $p|_{M^*} \vdash R_\phi(x)$ for any $|N|^+$-saturated $M \prec M^* \prec M'$).*

Proof. In the proof of the existence of honest definitions [CS12, Proposition 1.1], we show that $p_0(x) \wedge P(x) \vdash \phi(x, b)$ for some small $p_0 \subseteq p$, which translates to $p(x) \vdash R_{\phi(x,b)}(x)$. $\square$

Given $p \in S^{\mathrm{inv}}(M', M)$ we define $p' = \left\{R_{\phi(x,b)}(x) : p \vdash R_{\phi(x,b)}(x)\right\}$. It is clearly a complete type over $S(M^{\mathrm{ext}})$ and does not depend on the choice of N as it was only used to define the language. Thus we can identify it with a global type $F_M(p) = \{\phi(x, b) \in L(M') : \phi(M, b) = \psi(x) \in p'\}$ finitely satisfiable in M.

Recall that given a global type $p(x)$ and a definable function f, one defines $f_*(p) = \{\phi(x) : \phi(f(x)) \in p\}$. If p is M-invariant and f is M-definable, then $f_*(p)$ is also M-invariant.

Proposition 3.10. *The map F_M satisfies the following properties:*

1. *$F_M(p)|_M = p|_M$.*
2. *F_M is a continuous retraction from $S^{\mathrm{inv}}(\mathbb{M}, M)$ onto $S^{\mathrm{fs}}(\mathbb{M}, M)$.*
3. *If f is an M-definable function, then $f_*(F_M(p)) = F_M(f_*(p))$.*

Proof. (1) Clear from the construction.

(2) It is continuous as $F_M^{-1}(\phi(x, b)) = \bigcup_{\psi(x,c) \in L(\mathbb{M})} \{\psi(x, c) \wedge P(x) \vdash \phi(x, b)\}$. Now assume that p is actually finitely satisfiable in M, and that $\phi(x, b) \in L(\mathbb{M})$ is such that $\phi(x, b) \in p$ and $\neg\phi(x, b) \in F_M(p)$. But then $\neg\phi(M, b) = R_{\psi(x)}(M) \in p'$. This means that there is some $\chi(x) \in p$ such that $\chi(x) \vdash R_{\psi(x)}(x)$. But as $\chi(x) \wedge \neg\phi(x, b) \in p$, by finite satisfiability there is some $a \in M$ with $a \models R_{\psi(x)}(x) \wedge \neg\phi(x, b)$ — a contradiction. Thus F_M is the identity on $S^{\mathrm{fs}}(\mathbb{M}, M)$.

(3) First observe that it is enough to show that $f_*(p') = (f_*(p))'$. By compactness and Proposition 3.9 there is some $M \subseteq B \subseteq M'$ such that $|B| = |N|$ and $p|_B \vdash p'$, $f_*(p)|_B \vdash (f_*(p))'$. Let a in M' realize $p|_B$, and let $b = f(a)$. Then $b \models f_*(p)|_B$, thus $b \models (f_*(p))'$. On the other hand, as $a \models p'$, it follows that $b \models f_*(p')$. □

In fact, [CS12, Proposition 1.1] implies the following more explicit statement:

Proposition 3.11. *$(N', M') \succ (N, M)$, saturated enough. Then for every $\phi(x) \in L(N)$ there are some $\psi(x), \psi'(x) \in L(M')$ such that $\psi(x) \subseteq \phi(x) \subseteq \psi'(x)$, $\psi(M) = \psi'(M) = \phi(M)$ and $\psi'(x) \setminus \psi(x)$ divides over M.*

Proof. Let $\phi(x)$ be given. The proposition gives us a formula $\psi(x) \in L(M')$ such that $\psi(M') \subseteq \phi(M')$ and moreover no M-invariant type in $S_L(M')$ satisfies $\phi(x) \setminus \psi(x)$. Applying the proposition again to $\neg\phi(x)$, we find some $\chi(x) \in L(M')$ such that $\chi(M') \subseteq \neg\phi(M')$ and no M-invariant type in $S_L(M')$ satisfies $\neg\phi(x) \setminus \chi(x)$. But then take $\psi'(x) = \neg\chi(x)$. As M' is a model, it follows that $\psi(x) \subseteq \phi(x) \subseteq \psi'(x)$ and that no M-invariant type in $S_L(M')$ satisfies $\psi'(x) \setminus \psi(x)$. By saturation of M', NIP and Fact 2.1(1) it follows that $\psi'(x) \setminus \psi(x)$ divides over M.

□

3.3. **Extracting the finitely satisfiable part of an invariant measure.** Now we extend this map F_M to measures.

Remark 3.12. A measure $\mu \in \mathfrak{M}(A)$ is invariant (finitely satisfiable) over $B \subseteq A$ if and only if every $p \in S(\mu)$ is invariant (finitely satisfiable) over B.

Proof. It is clear that if μ is invariant (finitely satisfiable) over B then every $p \in S(\mu)$ is invariant (finitely satisfiable) over B. Conversely, assume that $\mu(\phi(x, a)) > 0$. Then it is easy to see by compactness that there is some $p \in S(\mu)$ with $\phi(x, a) \in p$. □

Assume that $\mu \in \mathfrak{M}(\mathbb{M})$ is M-invariant. Then we can define a measure μ' on $S^{\text{inv}}(\mathbb{M}, M)$ by setting $\mu'\left(S^{\text{inv}}(\mathbb{M}, M) \cap \phi(x, a)\right) = \mu(\phi(x, a)) / \mu\left(S^{\text{inv}}(\mathbb{M}, M)\right)$. If $\mu(\phi(x, a) \triangle \psi(x, b)) > 0$ then there is some $p \in S(\mu)$ with $\phi(x, a) \triangle \psi(x, b) \in p$. By the previous remark p is M-invariant, thus $\phi(x, a) \cap S^{\text{inv}}(\mathbb{M}, M) \neq \psi(x, b) \cap S^{\text{inv}}(\mathbb{M}, M)$, which implies that μ' is a well-defined.

Conversely, given a measure μ' on $S^{\text{inv}}(\mathbb{M}, M)$ we define

$$\mu(\phi(x, a)) = \mu'\left(\phi(x, a) \cap S^{\text{inv}}(\mathbb{M}, M)\right).$$

Then μ is a measure on $S(\mathbb{M})$, and every type in the support of μ is invariant, thus μ is invariant.

Remark 3.13. An M-invariant (resp. finitely satisfiable) measure $\mu \in \mathfrak{M}(\mathbb{M})$ is the same thing as a measure on $S^{\text{inv}}(\mathbb{M}, M)$ (resp. $S^{\text{fs}}(\mathbb{M}, M)$).

Definition 3.14. Let (X_1, Σ_1), (X_2, Σ_2) be measurable spaces and let a Borel mapping $f : X_1 \to X_2$ be given (e.g. a continuous map). Then, given a measure $\mu : \Sigma_1 \to [0, 1]$, the pushforward of μ is defined to be the measure $f_* (\mu) : \Sigma_2 \to [0, 1]$ given by $(f_* (\mu)) (A) = \mu \left(f^{-1} (A)\right)$ for $A \in \Sigma_2$.

Given an M-invariant global measure μ, by Remark 3.13 we view it as a measure μ' on the space of invariant types $S^{\mathrm{inv}} (\mathbb{M}, M)$. By Fact 3.14 and continuity of F_M we thus get a push-forward measure $(F_M)_* (\mu')$ on the space $S^{\mathrm{fs}} (\mathbb{M}, M)$. Again by Remark 3.13 this determines a measure ν on $S (\mathbb{M})$ which is finitely satisfiable in M. We define $F_M (\mu) = \nu$.

Proposition 3.15. *The map F_M satisfies the following properties:*

(1) $F_M (\mu) |_M = \mu|_M$.
(2) *F_M is a continuous retraction from $\mathfrak{M}^{\mathrm{inv}} (\mathbb{M}, M)$ to $\mathfrak{M}^{\mathrm{fs}} (\mathbb{M}, M)$.*
(3) *If f is an M-definable function, then $f_* (F_M (\mu)) = F_M (f_* (\mu))$.*

Proof. Follows from Proposition 3.10 by unwinding the definition of $F_M (\mu)$. □

3.4. **Lifting measures to Shelah's expansion.** The following fact is well-known for types, and we observe that it easily generalizes to measures.

Proposition 3.16. *Let T be* NIP*. Then measures on M^{ext} are in a natural one-to-one correspondence with global measures finitely satisfiable in M.*

Proof. By quantifier elimination, every definable subset of M^{ext} is of the form $\phi (M, a)$ for some $a \in \mathbb{M}$. Given a global measure μ finitely satisfiable in M, we define a measure $\mu' \in \mathfrak{M} (M')$ as follows: given an externally definable set $X \subseteq M$, we set $\mu' (X) = \mu (\phi (x, a))$ for some $\phi (x, a) \in L (\mathbb{M})$ such that $X = \phi (M, a)$. It is well-defined because if $X = \phi (M, a) = \psi (M, b)$ then $\mu (\phi (x, a)) = \mu (\psi (x, b))$ (as otherwise $\mu (\phi (x, a) \triangle \psi (x, b)) > 0$, thus there is some $c \models \phi (x, a) \triangle \psi (x, b)$ in M by finite satisfiability — a contradiction) and is clearly a measure on $S (M^{\mathrm{ext}})$.

Conversely, given a measure $\mu' \in \mathfrak{M} (M^{\mathrm{ext}})$, for $\phi (x, a) \in L (\mathbb{M})$ we define $\mu (\phi (x, a)) = \mu' (\phi (M, a))$. It is easy to see that μ is a global measure and that whenever $\mu (\phi (x, a)) > 0$ then $\mu' (\phi (M, a)) > 0$, thus $\phi (M, a)$ is non-empty. □

We are ready to prove the main theorem of the section.

Theorem 3.17. *Assume that T is* NIP*, $M \models T$ and G is an M-definable group.*

(1) *Let μ be a $G (M)$-invariant measure on $S_G (M)$. Then there is some measure μ' on $S_G (M^{\mathrm{ext}})$ which extends μ and is $G (M)$-invariant.*
(2) *Assume that the action of $G (M)$ on $S_G (M)$ has a fixed point p. Then there is some $p' \in S_G (M^{\mathrm{ext}})$ which extends p and is $G (M)$-invariant.*

Proof. Let $\mu \in \mathfrak{M} (M)$ be a $G (M)$-invariant measure on $S_G (M)$. By Proposition 3.5 there is some global measure μ' invariant over M which is in addition $G (\mathbb{M})$-invariant. Now let $\nu = F_M (\mu')$ be a global measure finitely satisfiable in M, as constructed in Section 3.2. By Proposition 3.15(1) it is still a measure on $S_G(\mathbb{M})$, extending μ. We claim that ν is $G(M)$-invariant. Indeed, by Proposition 3.15(3) and $G(M)$-invariance of μ', for every $g \in G(M)$ we have $g \cdot \nu = g \cdot F_M(\mu') = F_M(g \cdot \mu') = F_M(\mu') = \nu$. So ν is $G(M)$-invariant and finitely satisfiable in M, thus by Proposition 3.16 it corresponds to a $G(M)$-invariant measure on $S_G(M^{\text{ext}})$, as wanted.

For the case of the existence of a fixed point in $S_G(M)$ the proof goes through by restricting to zero-one measures. □

We remark that as both existence of a fixed point and definable amenability are properties of the theory, the same holds in the monster model of $\text{Th}(M^{\text{ext}})$.

3.5. **Definable f-generics and fsg.** We aim towards proving Theorem 3.19. We assume that T has NIP and $M \models T$. We begin by pointing out that any definable complete type over M has a unique extension to a complete type over M^{ext}. This was observed in Claim 1, Proposition 57, of [CS], but we give another proof here. We will use the notation at the beginning of Subsection 2.2, namely M, N, M', N', P, L'. In particular M as an L'-structure is precisely M^{ext}, and M' as an L'-structure is a saturated model of $Th(M^{\text{ext}})$, whose L-reduct can be identified with the monster model of T.

Lemma 3.18. *Suppose $p(x) \in S(M)$ is definable. Then $p(x)$ implies a unique complete type $p^*(x) \in S(M^{\text{ext}})$. Moreover if $\bar{p}$ is the unique heir of p over the L-structure M', then again $\bar{p}$ implies a unique complete type over M' as an L'-structure, which is precisely the unique heir of p^*.*

Proof. Let $\bar{p}$ be the unique heir of p over M' (as an L-structure). By Proposition 3.9, $\bar{p}$ implies a unique complete type $p^*(x)$ over M^{ext}. So if R_ϕ is in $p^*(x)$, then in a saturated elementary extension of $\left(N', M', (R_\phi)_{\phi \in L(N)}\right)$ we have the implication $\bar{p}(x) \wedge \mathbf{P}(x) \vdash R_\phi(x)$, so by compactness there is $\psi(x, c) \in \bar{p}$ such that $\left(N', M', (R_\phi)_{\phi \in L(N)}\right) \models \forall x \in \mathbf{P}(\psi(x, c) \to R_\phi(x))$
Let $\chi(y)$ be an L-formula over M which is the $\psi(x, y)$-definition of $\bar{p}$ (equivalently of p). Hence $\models \chi(c)$, so by Tarski-Vaught, there is $c_0 \in M$ such that $\left(N, M, (R_\phi)_{\phi \in L(N)}\right) \models \chi(c_0) \wedge \forall x \in \mathbf{P}(\psi(x, c_0) \to R_\phi(x))$

As $\psi(x, c_0) \in p(x)$, we see that $p(x)$ implies $R_\phi(x)$ as required.

This proves the first part of the Lemma.

The moreover clause follows in a similar fashion. Namely by the first part, $\bar{p}$, being definable, implies a unique complete type over $(M')^{\text{ext}}$, in particular implies a unique complete L'-type over M', which can be checked to be the unique heir of p^*. □

Theorem 3.19. *Suppose T is* NIP*, $M \models T$ and G is a group definable over M.*

(1) *If G has a global* fsg *type (with respect to M), then G has a global* fsg *type with respect to M^{ext} in* $\mathrm{Th}\left(M^{\mathrm{ext}}\right)$.
(2) *If G has a global f-generic which is definable over M, then the same is true for* $\mathrm{Th}\left(M^{\mathrm{ext}}\right)$.

Proof. (1) Let L'' be the language of $(M')^{\mathrm{ext}}$, and let M'' be a saturated elementary extension of M' as an L''-structure. As G is fsg in T and $M'' \succ_L M$, there is some $p \in S_L(M'')$ such that gp is finitely satisfiable in M for all $g \in G(M'')$. It determines a complete type $q \in S_{L''}((M')^{\mathrm{ext}})$ such that moreover gq is finitely satisfiable in M for all $g \in G(M')$. Let $r = q \restriction L'$, it satisfies the same property. As $M' \succ_{L'} M^{\mathrm{ext}}$ is a saturated extension, it follows that G is fsg in $\mathrm{Th}(M^{\mathrm{ext}})$.

(2) We continue with the same notation. Our assumptions give us a complete L-type $\bar{p}$ over M', which is definable over M and such that for every $g \in G(M')$, $g\bar{p}$ is definable over M. So the stabilizer of $\bar{p}$ is $G^{00}(M')$. By Lemma 3.18, $\bar{p}$ extends to a unique complete L'-type $\bar{p}^*$ over M' which is moreover definable over M. So $\mathrm{Stab}(\bar{p}^*)$ is also $G^{00}(M')$, in particular has bounded index, so clearly $\bar{p}^*$ is also a global f-generic of G, definable over M^{ext} in $Th(M^{\mathrm{ext}})$, as required. □

4. Connected components

In this section we will show that the model-theoretic connected components are not affected by adding externally definable sets. For simplicity of notations we will be assuming that our group G is the whole universe.

Let $N \succeq M \models T$. By an elementary pair of models (N, M) we always mean a structure in the language $L_{\mathbf{P}} = L \cup \{\mathbf{P}(x)\}$ whose universe is N and such that $\mathbf{P}(N) = M$. We say that an $L_{\mathbf{P}}$-formula is *bounded* if it is of the form $Q_0 x_0 \in \mathbf{P} \ldots Q_{n-1} x_{n-1} \in \mathbf{P}\, \phi(x_0, \ldots, x_{n-1}, \bar{y})$ where $Q_i \in \{\exists, \forall\}$ and $\phi(\bar{x}, \bar{y}) \in L$. We will denote the set of all bounded formulas by $L_{\mathbf{P}}^{\mathrm{bdd}}$.

An $L_{\mathbf{P}}$-formula $\phi(x, y) \in L_{\mathbf{P}}$ is NIP over $\mathbf{P}$ (modulo some fixed theory of elementary pairs $T_{\mathbf{P}}$) if for some $n < \omega$ there are no $(b_i : i < n)$ in N and $(a_s : s \subseteq n)$ in $\mathbf{P}$ such that $\phi(a_s, b_i) \Leftrightarrow i \in s$. By the usual compactness argument, $\phi(x, y)$ is not NIP over $\mathbf{P}$ if and only if there is some $(N, M) \models T_{\mathbf{P}}$ in which we can find an $L_{\mathbf{P}}$-indiscernible sequence $(a_i : i < \omega)$ in $\mathbf{P}$ and $b \in N$ such that $(N, M) \models \phi(a_i, b)$ $\Leftrightarrow i$ is even (any sufficiently saturated pair would do).

Remark 4.1. Let (N, M) be an elementary pair of models of an NIP theory T. Then every bounded formula is NIP over $\mathbf{P}$ modulo $T_{\mathbf{P}} = \mathrm{Th}(N, M)$.

Proof. Let $\phi(x, y) \in L_{\mathbf{P}}^{\mathrm{bdd}}$ be given, and assume that it is not NIP over $\mathbf{P}$. By the previous paragraph this means that there is some $(N, M) \models T_{\mathbf{P}}$, $(a_i : i < \omega)$ in M and $b \in N$ such that $(N, M) \models \phi(a_i, b) \Leftrightarrow i$ is even. Take some $M' \succ^+ M$. By Shelah's theorem $\mathrm{Th}(M^{\mathrm{ext}})$ eliminates quantifiers, that is for every $a \in N$ there is some $\psi(x,y) \in L$ and $b \in M'$ such that $\phi(M,a) = \psi(M,b)$. In particular $M' \models \psi(a_i,b) \Leftrightarrow i$ is even, contradicting the assumption that $T = \mathrm{Th}_L(M')$ is NIP. $\square$

Note that the theory $T_{\mathbf{P}}$ of pairs need not be NIP in general. In [CS12, Section 2] it is shown that if every $L_{\mathbf{P}}$ formula is equivalent to a bounded one, then $T_{\mathbf{P}}$ is NIP.

4.1. G^0**.** We begin with the easiest case. Let $N \succ M$ be saturated, of size bigger than $\left|2^M\right|^+$.

First we generalize some basic NIP lemmas to the case of externally definable subgroups.

Lemma 4.2. *For any formula $\phi(x,y)$ and integer n there is some k such that: there are $b_0,\dots,b_{k-1} \in N$ such that $\phi(M,b_i)$ is a subgroup of index $\leq n$ for each $i<k$ and for any $b \in N$, if $\phi(M,b)$ is a subgroup of index $\leq n$ then $\bigcap_{i<k}\phi(M,b_i) \subseteq \phi(M,b)$.*

Proof. The usual proof of the Baldwin-Saxl lemma goes through showing that there is some k such that any finite intersection $\bigcap_{i<n}\phi(M,b_i)$ is equal to a subintersection of size $\leq k$. As there are at most $\left|2^M\right|$ different externally definable subgroups of M, by compactness and saturation of N we can thus find $b_0,\dots,b_0 \in N$ as wanted. $\square$

Definition 4.3. Let $G^M_{\phi,n}$ be the intersection of all externally definable subgroups of M of index $\leq n$, of the form $\phi(M,b)$ for some external parameter b.

By the previous lemma it follows that there is some k and $(b_i : i<k)$ from N such that $G^M_{\phi,n} = \bigcap_{i<k}\phi(M,b_i)$.

Lemma 4.4.
(1) *$G^M_{\phi,n}$ is* $\mathrm{Aut}(M)$*-invariant.*
(2) *$G^M_{\phi,n}$ and $G^{M'}_{\phi,n}$ have the same index for any two saturated models M,M' (and it is bounded by kn).*

Proof. (1) Note that every $\sigma \in \mathrm{Aut}(M)$ extends to an automorphism σ' of the pair (N,M) (indeed, σ is a partial automorphism of N, thus extends to an automorphism σ' of N, which in particular fixes M setwise). We conclude as $a \in \phi(M,b) \Leftrightarrow \sigma(a) \in \phi(M,\sigma'(b))$ and $\phi(M,\sigma'(b))$ is still of finite index in M.

(2) The previous lemma gives the same upper bound k for any model M as it only depends on the VC-dimension of ϕ in models of T, hence the bound on the index.

Second, note that if $M \prec M'$ then the index can only go up. We show that it doesn't. Let $\psi(M',b)$ be an externally definable subgroup of M' of bounded index, say of index l. Then we add a new predicate R naming it. We see that $M' \models \left\{\forall x_0,\dots,x_n \in R \forall x'_0,\dots,x'_n \notin R \exists y \bigwedge_{i<n}\left(\psi(x_i,y) \wedge \neg\psi(x'_i,y)\right)\right\}_{n\in\omega}$ $\cup\{R$ is a subgroup of index $l\}$. By resplendence we can expand M to a model of the

same sentences. It thus follows by compactness that R is an externally ψ-definable subgroup of M of index l. Now applying this observation to $\psi(x, y_0, \ldots, y_{k-1}) = \bigwedge_{i<k} \phi(x, y_i)$ and $l = kn$ we can conclude. □

Theorem 4.5. $G^0(M) = G^0(M^{\text{ext}})$.

Proof. So we have $M \prec N$ and $G^M_{\phi,n} = \bigcap_{i<k} \phi(M, b_i)$ with $b_i \in N$ for $i < N$. Let $(N', M') \succ (N, M)$ be a saturated extension of the pair. Observe that $\bigcap_{i<k} \phi(M', b_i)$ has the same index in M' as $\bigcap_{i<k} \phi(M, b_i)$ in M by elementarity. On the other hand, $G^{M'}_{\phi,n}$ has the same index in M' as $G^M_{\phi,n}$ in M by Lemma 4.4. Thus $\bigcap_{i<k} \phi(M', b_i) \supseteq G^{M'}_{\phi,n}$ and their indexes are equal. This implies that $\bigcap_{i<k} \phi(M', b_i) = G^{M'}_{\phi,n}$. But as $G^{M'}_{\phi,n}$ is $\mathrm{Aut}_L(M')$-invariant (by Lemma 4.4) and definable in a saturated structure (N', M'), it follows by compactness that it is actually definable in M', and thus again by elementarity $G^M_{\phi,n}$ is definable in M.

As $G^0(M^{\text{ext}}) = \bigcap_{n<\omega, \phi \in L} G^M_{\phi,n}$ and every $G^M_{\phi,n}$ is of finite index, it follows that $G^0(M^{\text{ext}}) = G^0(M)$. □

Corollary 4.6. *Let $M \models T$ be arbitrary. Any externally definable subgroup of M of finite index is definable.*

Proof. By Theorem 4.5 any externally definable subgroup of M of finite index contains a definable subgroup of finite index. Hence it is a union of finitely many cosets of that subgroup, and thus M-definable. □

4.2. **Type-definable groups, invariant groups and bounded index in non-saturated models.** In our proofs for relative G^{00} and G^{∞} we will be using non-saturated models, so we prefer to make it precise what we will mean by "type-definable", "invariant" and "bounded index" etc. in this situation.

Definition 4.7. Let M be a model, and let $\mathbf{\Sigma}(x)$ be a disjunction of complete types over a subset of M.

(1) We say that $\mathbf{\Sigma}(x)$ is a *hereditary subgroup* of M (of some M-definable set $D(M)$) if $\Sigma(N)$ is a subgroup of N (resp. of $D(N)$) for every $N \succ M$.
(2) If $\mathbf{\Sigma}(x)$ is a hereditary subgroup of $D(M)$, we say that it is of *hereditarily bounded index* if for every saturated $N \succ M$ the group $\mathbf{\Sigma}(N)$ is of bounded index in $D(N)$, i.e. the index is less than the saturation of N.

The following two lemmas are immediate by compactness.

Lemma 4.8. *Let M be a model and $\mathbf{\Sigma}(x)$ a disjunction of complete types over a small set $A \subseteq M$ concentrating on a definable set $D(x)$.*

(1) *Assume that:*
   (a) *For every $p(x), q(x) \in \mathbf{\Sigma}(x)$ and every sequence of formulas $\bar{\phi} = (\phi_r(x))_{r(x) \in \mathbf{\Sigma}}$ with $\phi_r(x) \in r(x)$ there are some $\psi_p(x) \in p, \psi_q(x) \in$*

$q, n \in \omega$ and $r_0, \ldots, r_{n-1}$ such that $\psi_p(x) \wedge \psi_q(y) \wedge D(x) \wedge D(y) \to \bigvee_{i<n} \phi_{r_i}(x \cdot y) \wedge D(x \cdot y)$.

(b) *For every $p(x) \in \mathbf{\Sigma}(x)$ the uniquely determined type $p\left(x^{-1}\right)$ is also in $\mathbf{\Sigma}(x)$.*

Then $\mathbf{\Sigma}(M)$ is a subgroup of $D(M)$.

(2) *If M is saturated and A is a small subset of M, then the converse holds.*

Lemma 4.9. *Assume that M is saturated, that $\mathbf{\Sigma}(x)$ is a small disjunction of complete types over a small set A, and that $\mathbf{\Sigma}(M)$ is a subgroup of $D(M)$. Then the following are equivalent:*

(1) *For every sequence of formulas $\bar{\phi} = (\phi_p)_{p \in \mathbf{\Sigma}}$ with $\phi_p(x) \in p$, there are some $\phi_0, \ldots, \phi_{n-1} \in \bar{\phi}$ and $m \in \omega$ such that for any pairwise different $(a_i)_{i<m}$ from $D(M)$ we have $a_i^{-1} a_j \models \bigvee_{k<n} \phi_k(x)$ for some $i < j < m$.*

(2) *$\mathbf{\Sigma}(M)$ is of bounded index in $D(\mathbb{M})$, that is $[D(\mathbb{M}) : \mathbf{\Sigma}(M)] < \bar{\kappa}$ where $\bar{\kappa}$ is the saturation of M.*

Remark 4.10. Note that if $N \succ M$ and $\mathbf{\Sigma}(x)$ consists of types over M, then Lemma 4.8(1) and Lemma 4.9(1) hold in N if and only if they hold in M. This means that:

- $\mathbf{\Sigma}(x)$ is a hereditary subgroup of $D(x)$ if and only if it satisfies Lemma 4.8(1) in M.
- $\mathbf{\Sigma}(x)$ has hereditarily bounded index in $D(x)$ if and only if it satisfies Lemma 4.9(1) in M.

Lemma 4.11. *T is a complete theory in the language L. Let $\mathbf{\Sigma}(x, \bar{y})$ be a collection of L-types.*

(1) *Assume we are given elementary pairs (N_i, M_i) and $\bar{b}_i \in N_i$ for $i \in \{0, 1\}$. Assume that $\mathrm{tp}_{L_{\mathbf{P}}^{\forall,\mathrm{bdd}}}\left(\bar{b}_0\right) = \mathrm{tp}_{L_{\mathbf{P}}^{\forall,\mathrm{bdd}}}\left(\bar{b}_1\right)$ (that is, they agree on all formulas of the form $\forall z_0 \ldots z_n \in \mathbf{P}\, \phi(z_0, \ldots, z_n, \bar{y})$ with $\phi \in L$ in the corresponding pairs). Assume that $\mathbf{\Sigma}\left(x, \bar{b}_0\right)$ is a hereditary subgroup of $\mathbf{P}(x)$ of hereditarily bounded index in the pair (N_0, M_0). Then $\mathbf{\Sigma}\left(x, \bar{b}_1\right)$ is a hereditary subgroup of $\mathbf{P}(x)$ of hereditarily bounded index in the pair (N_1, M_1).*

(2) *Assume that we are given elementary pairs (N_i, M_i) such that $(M_i : i < \kappa)$ and $(N_i : i < \kappa)$ are L-elementary chains. Assume that $\bar{b} \in N_0$ is such that:*
   (a) *$\mathrm{tp}_{L_{\mathbf{P}}^{\forall,\mathrm{bdd}}}\left(\bar{b}\right)$ evaluated in (N_i, M_i) is constant for all i,*
   (b) *$\mathbf{\Sigma}\left(x, \bar{b}\right)$ is a hereditary subgroup of $\mathbf{P}(x)$ of hereditarily bounded index in the pair (N_0, M_0).*

   Let $M = \bigcup_{i<\kappa} M_i, N = \bigcup_{i<\kappa} N_i$. Then $\mathrm{tp}_{L_{\mathbf{P}}^{\forall,\mathrm{bdd}}}\left(\bar{b}\right)$ in (N, M) is the same as in (N_0, M_0), in particular $\mathbf{\Sigma}\left(x, \bar{b}\right)$ is a hereditary subgroup of $\mathbf{P}(x)$ of hereditarily bounded index in the pair (N, M).

Proof. (1) In view of the Remark 4.10 we have to check that Lemma 4.8(1) and Lemma 4.9(1) hold for $\mathbf{\Sigma}\left(x,\bar{b}_1\right)$ in (N_1, M_1). But this follows directly from $\mathrm{tp}_{L_{\mathbf{P}}^{\forall,\mathrm{bdd}}}\left(\bar{b}_0\right) = \mathrm{tp}_{L_{\mathbf{P}}^{\forall,\mathrm{bdd}}}\left(\bar{b}_1\right)$ and the assumption on $\mathbf{\Sigma}\left(x,\bar{b}_0\right)$.

(2) In view of (1) and the assumption it is enough to show that $\mathrm{tp}_{L_{\mathbf{P}}^{\forall,\mathrm{bdd}}}\left(\bar{b}\right)$ is the same in (N,M) as in some/any (N_i, M_i) for $i<\kappa$. Let $\psi\left(\bar{y}\right) = \forall a_0 \ldots a_{n-1} \in \mathbf{P}\,\phi\left(a_0,\ldots,a_{n-1},\bar{y}\right) \in L_{\mathbf{P}}^{\forall,\mathrm{bdd}}$ be given. Assume that $(N,M) \models \neg\phi\left(a_0,\ldots,a_{n-1},\bar{b}\right)$ with $\phi \in L$ and a_i from $M = \mathbf{P}\left(N\right)$. It follows by construction that there is some $\alpha < \kappa$ such that $a_0,\ldots,a_{n-1}$ are in M_α. As $N \succ_L N_\alpha$, we have that $(N_\alpha, M_\alpha) \models \neg\phi\left(a_0,\ldots,a_{n-1},\bar{b}\right)$, i.e. $(N_\alpha, M_\alpha) \models \neg\psi(\bar{b})$. And the converse is clear. □

4.3. G^{00}**.** Let T be an NIP theory in the language L, $M \models T$. Let (N,M) be a saturated elementary pair.

Definition 4.12. We consider all subgroups of $\mathbf{P}\left(N\right) = M$ of bounded index (that is of index less than the saturation) and definable as $\mathbf{\Sigma}\left(M,B\right)$ where B is a small tuple from N and $\mathbf{\Sigma}$ is a partial L'-type over B (where $L \subseteq L' \subseteq L_{\mathbf{P}}$). Let $G_{L'(B)}^{00}\left(N,M\right)$ be defined as the intersection of all such groups, and let $G_{L'}^{00}\left(N,M\right) = \bigcap_{B\subset N,\mathrm{small}} G_{L'(B)}^{00}\left(N,M\right)$.

The following is standard.

Fact 4.13. *Let (N,M) be a saturated pair, and let $\Sigma\left(x,\bar{y}\right)$ be a partial type. Let $\bar{b}$ be from N, and assume that $\Sigma\left(M,\bar{b}\right)$ is $\mathrm{Aut}_L\left(M\right)$-invariant. Then $\Sigma\left(M,\bar{b}\right)$ is definable by a partial L-type over $\emptyset$.*

Proof. Let $S = \left\{\mathrm{tp}_L\left(a\right) : a \in \Sigma\left(M,\bar{b}\right)\right\}$. By invariance it follows that $p\left(x\right) \wedge \mathbf{P}\left(x\right) \to \Sigma\left(x,\bar{b}\right)$ holds in (N,M) for every $p \in S$, which by saturation implies that there is some $\phi_p\left(x\right) \in p$ such that $\phi_p\left(x\right) \wedge \mathbf{P}\left(x\right) \to \Sigma\left(x,\bar{b}\right)$. As S is small, again by saturation it follows that $\bigvee_{i<n}\phi_{p_i}\left(x\right) \wedge \mathbf{P}\left(x\right) = \Sigma\left(x,\bar{b}\right) \wedge \mathbf{P}\left(x\right)$ for some $p_0,\ldots,p_{n-1}$ from S. □

First we observe existence of G^{00} relatively to $\mathbf{P}\left(x\right)$.

Proposition 4.14. *Let (N,M) be a saturated pair.*

1. *For any small set $B \subset N$, we have $G_{L_{\mathbf{P}}(\emptyset)}^{00}\left(N,M\right) \subseteq G_{L(B)}^{00}\left(N,M\right)$.*
2. *In particular it follows that $G_L^{00}\left(N,M\right) = G_{L(B')}^{00}\left(N,M\right)$ for some small $B' \subset N$, and $\left[M : G_L^{\infty}\left(N,M\right)\right] \leq 2^{2^{|T|}}$.*

Proof. All the formulas of the form $\phi\left(x,b\right) \wedge \mathbf{P}\left(x\right)$ with $\phi\left(x,y\right) \in L$ are NIP. Then the usual proof of the existence of G^{00} in NIP theories, see e.g. [HPP08, Proposition 6.1], goes through unchanged and gives that for a subgroup of $M = \mathbf{P}\left(N\right)$ of the form $\Sigma\left(M,\bar{b}\right)$ where $\Sigma\left(x,y\right)$ is a partial L-type, of bounded index, there are only boundedly many different conjugates of it under $L_{\mathbf{P}}$-automorphisms. Then the proposition follows by taking the intersection of all such conjugates, over all $L_{\mathbf{P}}$-types of $|\bar{y}|$-tuples over $\emptyset$, which is still of bounded index.

To see the absolute bound on the index, note that $G^{00}_L(N,M)$ is invariant under $L_{\mathbf{P}}$-automorphisms and type-definable, so it follows by saturation that it is $L_{\mathbf{P}}$-type-definable over $\emptyset$. Then the bound follows by the usual application of Erdős-Rado. □

Now we will show that G^{00} is not changed by adding externally definable sets.

Theorem 4.15. *Let (N,M) be a saturated pair. Then $G^{00}(M)=G^{00}_L(N,M)$.*

Proof. Fix a cardinal $\lambda \gg 2^{2^{|T|}}$. By induction on $\alpha \leq \lambda$ we try to define M_α, N_α, $\mathbf{\Sigma}_\alpha(x,\bar{y}_\alpha)$, $\bar{b}_\alpha$ such that:

1. $(M_\alpha)_{\alpha\leq\lambda}$ and $(N_\alpha)_{\alpha\leq\lambda}$ are L-elementary chains of models;
2. (N_α, M_α) is a saturated elementary pair;
3. $\mathbf{\Sigma}_\alpha(x,\bar{y}_\alpha)$ is a partial L-type of size bounded with respect to the saturation of (N_α,M_α);
4. $\bar{b}_\alpha \in N_\alpha$;
5. For any $\alpha \leq \lambda$, $\mathrm{tp}_{L^{\forall,\mathrm{bdd}}_{\mathbf{P}}}\left((\bar{b}_i)_{i\leq\alpha}\right)$ is the same evaluated in any of the pairs (N_β,M_β) for all $\alpha\leq\beta\leq\lambda$;
6. For each $\alpha\leq\lambda$, $\mathbf{\Sigma}_\alpha\left(x,\bar{b}_\alpha\right)$ is a hereditary subgroup of $\mathbf{P}(x)$ of hereditarily bounded index in (N_α,M_α);
7. $\mathbf{\Sigma}_i(M_\alpha,\bar{b}_i)\subsetneq\mathbf{\Sigma}_j(M_\alpha,\bar{b}_j)$ for all $j<i\leq\alpha<\lambda$;
8. $M_0=M, N_0=N$ and $\mathbf{\Sigma}_0(M_0,\bar{b}_0)=G^{00}_L(N,M)$.

Suppose that we manage to carry out the induction. But then this means that in a saturated pair (N_λ,M_λ) we have a strictly decreasing sequence $\left(\mathbf{\Sigma}(M_\lambda,\bar{b}_i):i<\lambda\right)$ of subgroups of M_λ of bounded index which are definable by small L-types over small sets of parameters from N_λ — contradicting Theorem 4.14.

So let α^* be the smallest ordinal at which we got stuck.

Claim 1: α^* is a successor.

Proof: Assume that $\alpha^*=\bigcup_{\alpha<\alpha^*}\alpha$ is a limit ordinal. We then define $M'=\bigcup_{\alpha<\alpha^*}M_\alpha$, and $N'=\bigcup_{\alpha<\alpha^*}N_\alpha$. Note that (N',M') is an elementary pair. Let $(N_{\alpha^*},M_{\alpha^*})$ be some saturated extension of (N',M'). Let $\bar{b}_{\alpha^*}=\bigcup_{\alpha<\alpha^*}\bar{b}_\alpha$ and $\mathbf{\Sigma}_{\alpha^*}(x,\bar{b}_{\alpha^*})=\bigcap_{\alpha<\alpha^*}\mathbf{\Sigma}_\alpha(x,\bar{b}_\alpha)$. Using Lemma 4.11(1),(2) and the inductive assumption it is easy to verify that $(N_{\alpha^*},M_{\alpha^*})$ and $\mathbf{\Sigma}_{\alpha^*}(x,\bar{b}_{\alpha^*})$ satisfy all the requirements (1)–(8). But this contradicts the choice of α^*.

So $\alpha^*=\alpha+1$ is a successor. Take $K\succ N_\alpha\succ M_\alpha$ very saturated. We let $\Lambda(M_\alpha)$ be the intersection of all hereditary subgroups of M_α of hereditarily bounded index, in the sense of the pair (K,M_α), definable by partial L-types with parameters from K. Note that this intersection might contain $2^{|M_\alpha|}$-many L-formulas, but that's ok. Note that $\bar{b}_{\leq\alpha}$ has the same $L^{\mathrm{bdd}}_{\mathbf{P}}$-type in (K,M_α) as in (N_α,M_α) (as it is determined by $\mathrm{tp}_L(\bar{b}_{\leq\alpha}/M_\alpha)$). In view of Lemma 4.11(1) we have $\Lambda(M_\alpha) \subseteq G_L^{00}(N_\alpha, M_\alpha) \subseteq \mathbf{\Sigma}_\alpha(M_\alpha, \bar{b}_\alpha)$.

Claim 2: $\Lambda(M_\alpha) = G_L^{00}(N_\alpha, M_\alpha)$.

Proof: Assume that $\Lambda(M_\alpha) \subsetneq G_L^{00}(N_\alpha, M_\alpha)$. Let $(N_{\alpha+1}, M_{\alpha+1})$ be a saturated elementary extension of (K, M_α) and set $\mathbf{\Sigma}_{\alpha+1}(x, \bar{b}_{\alpha+1}) = \Lambda(x)$ (which is defined an intersection of size $\leq 2^{|M_\alpha|}$ of hereditary subgroups of hereditarily bounded index, thus is of hereditarily bounded index and is relatively definable by an L-type in $(N_{\alpha+1}, M_{\alpha+1})$). It is now easy to see using the inductive assumption and Lemma 4.11(1) that all the conditions (1)–(8) are satisfied for $\alpha^* = \alpha+1$, which means that we could have continued the induction contradicting the choice of α, so the claim is proved.

As every L-automorphism of M_α extends to an $L_{\mathbf{P}}$-automorphism of the pair (K, M_α) by saturation of K, it follows that $\Lambda(M_\alpha)$ is $\mathrm{Aut}_L(M_\alpha)$-invariant. But by Claim 2 this means that $G_L^{00}(N_\alpha, M_\alpha)$ is an $\mathrm{Aut}_L(M_\alpha)$-invariant subgroup of M_α. Along with Fact 4.13 and (7) this implies that $G^{00}(M_\alpha) \subseteq G_L^{00}(N_\alpha, M_\alpha) \subseteq \mathbf{\Sigma}_\alpha\left(M_\alpha, \bar{b}_\alpha\right) \subseteq \mathbf{\Sigma}_0(M_\alpha, \bar{b}_0)$. As $G^{00}(M_\alpha)$ is $\bigwedge$-definable over $\emptyset$, we have $G^{00}(M_0) \subseteq \mathbf{\Sigma}_0(M_0, b_0) = G_L^{00}(N_0, M_0)$ — as wanted.

∎

Corollary 4.16. *Let T be a theory in the language L, $M \models T$. Let $M \models T$, and let $\widetilde{M} \succ M^{\mathrm{ext}}$ be a monster model for Shelah's expansion of M, in the language L'. Then $G^{00}\left(\widetilde{M}\right) = G^{00}\left(\widetilde{M} \restriction L\right)$.*

Proof. Let $N \succ^+ M$ and, let $(N', M') \succ^+ (N, M)$. We may identify $\widetilde{M}$ with M', in such a way that every $\phi(\bar{x}) \in L'(\emptyset)$ is equivalent on M' to some $\psi(\bar{x}) \in L(N)$. Consider $G^{00}(M')$, as $\mathrm{Th}_{L'}(M')$ is NIP by Shelah's theorem, it follows from existence of G^{00} in NIP theories that $G^{00}(M')$ is definable by a partial L'-type over $\emptyset$, thus definable by a partial L-type over N. That is, $G^{00}\left(\widetilde{M}\right) \supseteq G_L^{00}(N', M')$, and we can conclude by Theorem 4.15. ∎

Corollary 4.17. *Let (N, M) be an elementary pair, and assume that $H(M)$ is a (hereditary) subgroup of M of hereditarily bounded index, which is $L(N)$-type-definable. Then it is $L(M)$-type-definable.*

Proof. Let (N', M') be a saturated extension of (N, M). By the theorem we know that $G^{00}(M') \subseteq H(M')$, and so $H(M')$ is a union of cosets of $G^{00}(M')$. As each coset has a representative in M, it follows that $H(M')$ is $\mathrm{Aut}_L(M'/M)$-invariant. By Fact 4.13 it follows that $H(M')$ is $L(M)$-type-definable, and so $H(M)$ as well. ∎

4.4. G^{∞}**.** Again let T be an NIP theory in the language L, $M \models T$. Let (N, M) be a saturated elementary pair.

Definition 4.18. We consider all subgroups of $\mathbf{P}(N) = M$ of bounded index (that is of index less than the saturation) and definable as $\mathbf{\Sigma}(M, B)$ where B is a small tuple from N and $\mathbf{\Sigma}$ is a disjunction of complete L'-types over B (where $L \subseteq L' \subseteq L_{\mathbf{P}}$). Let $G^{\infty}_{L'(B)}(N, M)$ be defined as the intersection of all such groups, and let $G^{\infty}_{L'}(N, M) = \bigcap_{B \subset N, \text{small}} G^{\infty}_{L'(B)}(N, M)$.

First we establish a version of the existence of G^{∞} relatively to a predicate $\mathbf{P}(x)$.

Theorem 4.19. *Let (N, M) be a saturated pair. Then:*

(1) *For any small set $B \subset N$, we have $G^{\infty}_{L_{\mathbf{P}}(\emptyset)}(N, M) \subseteq G^{\infty}_{L^{\text{bdd}}_{\mathbf{P}}(B)}(N, M)$.*
(2) *In particular it follows that $G^{\infty}_{L}(N, M) = G^{\infty}_{L(B')}(N, M)$ for some small $B' \subset N$, and $[M : G^{\infty}_{L}(N, M)] \leq 2^{2^{|T|}}$.*

Proof. Let $\bar{\kappa}$ be the saturation of the pair (a strong limit, of large enough cofinality).

For a small set $A \subseteq N$, we define $X_A = \left\{a^{-1}b : a, b \in \mathbf{P} \wedge a \equiv^{L^{\text{bdd}}_{\mathbf{P}}}_{A} b\right\}$. As usual, given sets X, Y, we denote $X^Y = \{x^y : x \in X, y \in Y\}$ where $x^y = y^{-1}xy$, and $X^n = \{x_1 x_2 \dots x_n : x_i \in X\}$.

Claim. $(X_A)^{\mathbf{P}} \subseteq (X_A)^2$

Proof: Let $a \equiv^{L^{\text{bdd}}_{\mathbf{P}}}_{A} b$ and c from $\mathbf{P}$ be arbitrary. Then we have $(X_A)^{\mathbf{P}} \ni \left(a^{-1}b\right)^c = (ac)^{-1} bc = \dots$ (by the assumption and compactness there is some $d \in \mathbf{P}$ such that $(a, c) \equiv^{L^{\text{bdd}}_{\mathbf{P}}}_{A} (b, d)$, so in particular $c \equiv^{L^{\text{bdd}}_{\mathbf{P}}}_{A} d$ and $ac \equiv^{L^{\text{bdd}}_{\mathbf{P}}}_{A} bd$)$\dots = \left((ac)^{-1}(bd)\right)\left(d^{-1}c\right) \in (X_A)^2$.

Assume that $G^{\infty}_{L_{\mathbf{P}}(\emptyset)}(N, M) \not\subseteq G^{\infty}_{L^{\text{bdd}}_{\mathbf{P}}(A)}(N, M)$ for some small $A \subseteq N$. Let $B \subseteq N, |B| \leq \lambda$ be a small set containing representatives of all cosets of all subroups of bounded index which are definable by disjunctions of complete $L^{\text{bdd}}_{\mathbf{P}}$-types over A. Let $G = \mathbf{\Sigma}(x, A)$ be a subgroup of $\mathbf{P}$ of bounded index. Let $a \equiv^{L^{\text{bdd}}_{\mathbf{P}}}_{B} b$ be arbitrary. By assumption there is some $c \in B$ from the same coset of G as a. It follows that b is from the same coset of G as c, thus as a. But this implies that $a^{-1}b \in G$. It follows that $\langle X_B \rangle \subseteq G^{\infty}_{L^{\text{bdd}}_{\mathbf{P}}(A)}(N, M)$. Let $X = \bigcap_{B' \subset N, |B'| \leq \lambda} \langle X_{B'} \rangle$. Note that X is invariant with respect to $L_{\mathbf{P}}$-automorphisms of the pair (over $\emptyset$). So, if X had bounded index in $\mathbf{P}$, we would have $G^{\infty}_{L_{\mathbf{P}}(\emptyset)}(N, M) \subseteq X \subseteq \langle X_B \rangle$, a contradiction.

Thus X has unbounded index in $\mathbf{P}$, and we can find arbitrary long sequences $(B_i, c_i)_{i \in \kappa}$ with $B_i \subset N, |B_i| \leq \lambda$ and $c_i \in \mathbf{P}$ such that $c_i \in \left(\bigcap_{j<i} \langle X_{B_j} \rangle\right) \setminus \langle X_{B_i} \rangle$. By Erdős-Rado we may find such a sequence which is moreover $L_{\mathbf{P}}$-indiscernible. In particular, for some $m \in \omega$ we have:

(1) $c_i \in \left(\bigcap_{j<i} X^m_{B_j}\right) \setminus X^{m+4}_{B_i}$ for all i.

Next, using $L_{\mathbf{P}}$-indiscernibility of the sequence, the claim and compactness we can find some finite sets of formulas $\Phi, \Phi' \subset L_{\mathbf{P}}^{\mathrm{bdd}}$ such that:

(2) $c_i \notin (X_{B_i,\Phi})^{m+4}$,
(3) $(X_{B_i,\Phi'})^{\mathbf{P}} \subseteq (X_{B_i,\Phi})^2$.

(where $X_{A,\Phi} = \left\{a^{-1}b : \bigwedge_{\phi\in\Phi} (\phi(a,A) \leftrightarrow \phi(b,A))\right\}$).

Now, for an arbitrary finite sequence $I = (i_1, \ldots, i_n)$ of pairwise distinct elements of ω define the following elements of $\mathbf{P}$:

- $c_{I,0} = c_{2i_1+1} \cdot \cdots \cdot c_{2i_n+1}$,
- $c_{I,1} = c_{2i_1} \cdot \ldots \cdot c_{2i_n}$

To obtain a contradiction it is sufficient to show:

(4) if $j \notin I$, then $c_{I,0}c_{I,1}^{-1} \in X_{B_{2j}} \subseteq X_{B_{2j},\Phi'}$
(5) if $j \in I$, then $c_{I,0}c_{I,1}^{-1} \notin X_{B_{2j},\Phi'}$

(as then the formula $\psi(x_1x_2,\bar{z}) = \bigwedge_{\phi\in\Phi'} (\phi(x_1,\bar{z}) \leftrightarrow \phi(x_2,\bar{z})) \in L_{\mathbf{P}}^{\mathrm{bdd}}$ would have IP over $\mathbf{P}$ in (N,M), witnessed by $\left(\left(c_{I,0}^{-1}, c_{I,1}^{-1}\right) : I \subset \omega\right)$ in $\mathbf{P}$ and $(B_{2j} : j < \omega)$ from N, contradicting Remark 4.1).

So if $j \notin I$, then $c_{I,0} \equiv_{B_{2j}}^{L_{\mathbf{P}}} c_{I,1}$ by $L_{\mathbf{P}}$-indiscernibility of our sequence, equivalently $c_{I,0}^{-1} \equiv_{B_{2j}}^{L_{\mathbf{P}}} c_{I,1}^{-1}$ thus $c_{I,0}c_{I,1}^{-1} \in X_{B_{2j}}$ and (4) follows.

Assume that (5) does not hold, then get a contradiction like in [Gis11, Theorem 5.3] using (3) and (2).

□

Proposition 4.20. *Let (N,M) be a saturated pair. Then $G_L^\infty(N,M) = G^\infty(M)$.*

Proof. We can repeat the proof of Theorem 4.15 using Theorem 4.19 instead of Theorem 4.14, but here is a shorter argument.

Let $K \succ^+ M$, and let $(K',M') \succ^+ (K,M)$. Let $\Lambda(M)$ be the intersection of all hereditary subgroups of M of hereditarily bounded index (in the sense of the pair (K,M)) definable by disjunctions of L-types with parameters from K. By saturation of K over M and Lemma 4.11(1) it follows that $\Lambda(M) \subseteq G_L^\infty(N,M)$.

Note that $G_L^\infty(K',M') \subseteq \Lambda(M')$, so by Theorem 4.19 we have $[M' : \Lambda(M')] \leq 2^{2^{|T|}}$, which implies $[M : \Lambda(M)] \leq 2^{2^{|T|}}$. Now $\Lambda(M)$ is an L-invariant subgroup of M (by saturation of K every L-automorphism of M extends to an $L_{\mathbf{P}}$-automorphism of (K,M)) and of index smaller than the L-saturation of M, so $G^\infty(M) \subseteq \Lambda(M) \subseteq G_L^\infty(N,M)$. □

Corollary 4.21. *Let T be a theory in the language L, $M \models T$. Let $M \models T$, and let $\widetilde{M} \succ M^{\mathrm{ext}}$ be a monster model for Shelah's expansion of M, in the language L'. Then $G^\infty\left(\widetilde{M}\right) = G^\infty\left(\widetilde{M} \upharpoonright L\right)$.*

Proof. Same as the proof of Corollary 4.16. □

Problem 4.22. (1) By [HP11, Remark 8.3] if G is a definably amenable NIP group such that G/G^{00} is a compact Lie group, then G^{00} is externally definable. Is there any generalization of this fact for arbitrary NIP groups, or at least for the finite dp-rank case? E.g., does naming G^{00} by a predicate preserve NIP?

(2) In view of the results of this section, one can try to understand various connected components and quotients in an elementary pair of models in terms of the base theory.

5. Topological dynamics and the "Ellis group" conjecture

5.1. **Topological dynamics and minimal flows.** The subject topological dynamics tries to understand a topological group via its actions on compact spaces. A good reference is [Aus88]. As originally suggested by Newelski, topological dynamics yields new insights into the model theory of definable groups, as well as new invariants, which are especially relevant to generalizing stable group theory to other "tame" contexts, such as groups in NIP theories. In [Pil13, GPP12b], a theory of "definable" topological dynamics was developed, following earlier work of Newelski.

The context is: a model M_0 and a group G (identified with its points in a saturated elementary extension of M_0) which is definable over M_0.

ASSUMPTION: All types in $S_G(M_0)$ are definable.

Two extreme cases are:

(a) M_0 is the standard model of set theory, and $G(M_0)$ is a group,

(b) T is an NIP theory, $M \models T$, G a group definable over M, and $M_0 = M^{\mathrm{ext}}$.

In case (a) our theory reduces to the classical topological dynamics of the discrete group $G(M_0)$. In case (b) which is the interest of the current paper, we at least obtain some new invariants and problems. We summarize the theory developed in [GPP12b], as background for the results of this section.

We call a map f from $G(M_0)$ to a compact space C *definable* if for any disjoint closed sets C_1, C_2 of C, $f^{-1}(C_1)$ and $f^{-1}(C_2)$ are separated by a definable set. An action of $G(M_0)$ on a compact space C (by homeomorphisms) is "definable" if for any $x \in C$, the map from $G(M_0)$ to C which takes $g \in G$ to gc is definable. Such actions are called definable $G(M_0)$-flows.

Fact 5.1. (1) *The left action of G on $S_G(M_0)$ is definable. Moreover $(S_G(M_0), 1)$ is the (unique) universal definable $G(M_0)$-ambit; where by a definable $G(M_0)$-ambit we mean a definable $G(M_0)$-flow X with a distinguished point x whose orbit is dense.*

(2) *$S_G(M_0)$ has a semigroup structure $\cdot$, which extends the group operation on $G(M_0)$ and is continuous in the first coordinate. For $p, q \in S_G(M_0)$, $p \cdot q$*

is $\mathrm{tp}(a \cdot b/M_0)$ *where* b *realizes* q *and* a *realizes the unique coheir of* p *over* M_0, b.

(3) *Left ideals of* $S_G(M_0)$ *are precisely closed* $G(M_0)$*-invariant subspaces (i.e. subflows of the definable* $G(M_0)$*-flow* $S_G(M_0)$*).*
(4) *There is a unique (up to isomorphism) minimal definable* $G(M_0)$*-flow* $\mathcal{M}$*, which coincides with some/any minimal subflow of* $S_G(M_0)$.
(5) *Pick a minimal subflow* $\mathcal{M}$ *of* $S_G(M_0)$ *and an idempotent* $u \in \mathcal{M}$*. Then* $u \cdot \mathcal{M}$ *is a subgroup of the semigroup* $S_G(M_0)$*, whose isomorphism type does not depend on the choice of* $\mathcal{M}$ *or* u*. We call* $u \cdot \mathcal{M}$ *the Ellis group attached to the data. It also has a certain compact* T_1 *topology, with respect to which the group structure is separately continuous, but this will not really concern us here.*
(6) *Using these ideas, the notions of definable amenability and definable extreme amenability can be characterized in a fashion similar to their characterization in the discrete case (e.g. a definable group* G *is definably extremely amenable if and only if every definable action of it has a fixed point).*

5.2. **Almost periodic types.**

Definition 5.2. A type $p \in S_G(M_0)$ is called *almost periodic* if the closure of the orbit (under $G(M_0)$) of p is a minimal $G(M_0)$-flow.

The usual characterization of almost periodicity holds:

Fact 5.3. *The following are equivalent for a type* $p \in S(M_0)$*:*

(1) $p(x)$ *is almost periodic.*
(2) *For every* $\phi(x) \in p$*, the set* $\overline{Gp}$ *is covered by finitely many left translates of* $\phi(x)$.
(3) *For every formula* $\phi(x) \in p$*,* $\{g \in G(M_0) : g\phi \in p\}$ *(which is a definable subset of* $G(M_0)$ *by definability of* p*) is right generic, namely finitely many right translates cover* $G(M_0)$.

Proof. The equivalence of (1) and (2) holds by e.g. [New09, Remark 1.6].

(3) $\Rightarrow$ (1): Suppose (3) holds. Suppose $q \in \overline{G(M_0)p}$. Let $\phi(x) \in p$. Let $Z = \{g \in G(M_0) : g\phi \in p\}$, so $Zg_1 \cup .. \cup Zg_n = G(M_0)$ for some $g_1, .., g_n \in G(M_0)$. Hence $g_1^{-1}\phi \vee ... \vee g_n^{-1}\phi \in gp$ for all $g \in G(M_0)$, whereby some $g_i^{-1}\phi \in q$, so $\phi \in g_i q$. We have shown that $p \in \overline{G(M_0)q}$. So $\overline{G(M_0)p}$ is minimal.
(1) $\Rightarrow$ (3): Suppose p is almost periodic, $\phi \in p$ and $Z = \{g \in G(M_0) : g\phi \in p\}$. Let $\mathcal{M} = \overline{G(M_0)p}$. Now by (2) there are $g_1, .., g_n \in G(M_0)$ such that the clopen set $g_1\phi \vee .. \vee g_n\phi$ includes $\mathcal{M}$. It follows from the definition of Z that $Zg_1^{-1} \cup ... \cup Zg_n^{-1} = G_0(M)$.

□

5.3. **The Ellis group conjecture.** The quotient map from G to $G/G^{00}_{M_0}$ factors through the tautological map $g \to \mathrm{tp}(g/M_0)$ from G to $S_G(M_0)$, and we let π denote the resulting map from $S_G(M_0)$ to $G/G^{00}_{M_0}$. It was pointed out in [GPP12b] that $G/G^{00}_{M_0}$ is the "universal definable compactification" of $G(M_0)$, which in case (a) from Section 5.1 is what is called the Bohr compactification of the discrete group $G(M_0)$.

Let us note:

Remark 5.4. The map π is a surjective semigroup homomorphism, and for any minimal subflow $\mathcal{M}$ of $S_G(M_0)$ and idempotent $u \in \mathcal{M}$, the restriction of π to $u \cdot \mathcal{M}$ is surjective, hence a surjective group homomorphism.

Proof. The only thing possibly requiring a proof is the surjectivity of the restriction of π to $u \cdot \mathcal{M}$. Let $g \in G$ and $p = \mathrm{tp}(g/M_0)$. Then $p \cdot u \in \mathcal{M}$ as the latter is a left ideal. Hence $u \cdot (p \cdot u) \in \mathcal{M}$ and as $\pi(u)$ is the identity of $G/G^{00}_{M_0}$ we see that $\pi(u \cdot p \cdot u) = \pi(p)$.

□

We now restrict to case (b) above: namely T is NIP, G is a group definable over a model $M \models T$ and $M_0 = M^{\mathrm{ext}}$. We make free use of the results from the previous sections, namely the preservation of various properties and objects associated to G (definable amenability, G/G^{00}, etc) when passing from T to $Th(M_0)$.

Ellis group conjecture. Suppose G is definably amenable. Then the restriction of $\pi : S_G(M_0) \to G/G^{00}$ to $u \cdot \mathcal{M}$ is an isomorphism (for some/any choice of minimal subflow $\mathcal{M}$ of $S_G(M_0)$ and idempotent $u \in \mathcal{M}$).

We remark that without the definable amenability assumption this statement is not true even for groups definable in o-minimal theories, see [GPP12a]. In the following subsections, we will prove (or explain) the announced cases of the conjecture, thus establishing Theorem 1.4.

5.4. G **is definably extremely amenable, proof of Theorem 1.4(1).** It was observed in Proposition 3.7 that if G is definably extremely amenable then $G = G^{00}$. On the other hand by definition of extreme amenability $u \cdot \mathcal{M} = \{u\}$ for any minimal subflow $\mathcal{M}$ and idempotent $u \in \mathcal{M}$.

5.5. G **is fsg, Theorem 1.4(2).** This was proved in [Pil13] when G is fsg namely has a global type every translate of which is finitely satisfiable in M. We summarize the situation for the sake of completeness:

Fact 5.5. *Let G be* fsg. *Then we have:*

(1) *A global type is left (equivalently, right) generic if and only if it is left (equivalently, right) f-generic over M_0, if and only if every translate is finitely satisfiable in M_0. In particular, the answer to Problem* **??** *is positive.*

(2) *There is a unique minimal subflow of $S_G(M_0)$, namely the set of generic types, and the Ellis group conjecture holds.*

5.6. G **admits a definable f-generic, proof of Theorem 1.4(3).** The other "extreme case" of definable amenability is when there is a global f-generic type, definable over M. We expect that if G has some global definable f-generic type, then there is one which is definable over M. This feature was also considered by Hrushovski in [Hru], under the name "groups with definable generics" and Example 6.30 of that paper gives several examples from the theory of algebraically closed valued fields.

Proposition 5.6. *Suppose G has a global f-generic type, definable over M_0. Then*
(i) $G^{00} = G^0$.
(ii) For $p \in S_G(M_0)$, p is almost periodic if and only if the global heir of p is f-generic.
(iii) Any minimal subflow $\mathcal{M}$ of $S_G(M_0)$ is already a group, so coincides with the Ellis group.
(iv) The Ellis group conjecture holds: the restriction of π to $\mathcal{M}$ is an isomorphism with G/G^0.

Proof. (i) Working in T, let q be a global f-generic definable over M (or just definable). By Fact 3.3, the left stabilizer of q is G^{00}. But this left stabilizer is clearly an intersection of M-definable subgroups: for each $\phi(x,y) \in L$, $\mathrm{Stab}_\phi(q) = \{g \in G : \phi(x,c) \in q \text{ iff } \phi(g^{-1}x,c) \in q \text{ for all } c\}$. So each $\mathrm{Stab}_\phi(p)$ is finite index, whereby $G^{00} = G^0$.

(ii) First assume that $p \in S_G(M_0)$ and that the global heir $\bar{p}$ of p is f-generic. We will use Fact 5.3. Let $\phi(x) \in p$. Then $X = \{g \in G : g\phi \in \bar{p}\}$ is definable over M_0 (by definability over M_0 of $\bar{p}$). Now X contains the left stabilizer of $\bar{p}$ which, by Fact 3.3, is G^{00}. As G^{00} has bounded index in G (and is a normal subgroup) and X is definable, finitely many *right* translates of X cover G. Hence as X is definable over M_0 the same thing is true in $G(M_0)$.

The converse is a little more complicated. First by 3.19, there is a global f-generic $\bar{p}$ of G with respect to $Th(M_0)$ which is definable over M_0. Let p be the restriction of $\bar{p}$ to M_0, so $\bar{p}$ is the unique global heir of p and by the first part of the proof, p is almost periodic. Let $I = \overline{G(M_0)p}$. We first note that for any $q \in I$ the global heir of q is f-generic. This is because $q = \mathrm{tp}(ab/M_0)$ where $a \in G$ and b realizes the unique heir of q over M_0, a by Fact 5.1. But then the unique global heir of q is precisely $a\bar{p}$ which we know to be f-generic and definable.

Now let $q \in S_G(M_0)$ be an almost periodic type, not necessarily in I. Let $J = \overline{G(M_0)q}$. By what we saw in the last paragraph, it suffices to show that some $r \in J$ has the required property. From material in Section 3 of [GPP12b], the map from I to J which takes $p' \in I$ to $p' \cdot q$ is an isomorphism of $G(M_0)$-flows, namely a homeomorphism which commutes with the action of $G(M_0)$. Let $r = p \cdot q$, and we show that r (or its global heir) is as required, which will be enough. We let L denote the the language of the structure M_0.

Claim. For any L-formula $\phi(x,y)$, $\mathrm{Stab}_\phi(r) = \{g \in G(M_0)$: for all $c \in M_0$, $\phi(x,c) \in r \leftrightarrow g\phi(x,c) \in r\}$ is a definable subgroup of $G(M_0)$ of finite index.

Granted the claim, let $\bar{r}$ be the unique global heir of r (i.e. defined by the same defining schema), and we see that $\mathrm{Stab}(\bar{r}) = G^0$. Hence r is a global f-generic type definable over M_0, and we are finished.

Proof of Claim. Definability is immediate, by definability of the type r. Let g_1 realize p, and a realize the unique heir of q over M_0, g_1, So $g_1 a$ realizes r. Let $g \in G(M_0)$, $c \in M_0$ and $\phi(x,y) \in L$. Let $\psi(z,y)$ be the $\phi(zx,y)$-definition for q. Then $g\phi(x,c) \in r$ iff $\phi(g^{-1}x,c) \in r$ iff $\models \phi(g^{-1}g_1 a, c)$ iff $\phi((g^{-1}g_1)x,c)$ is in the unique heir of q over M_0, g_1 iff $\models \psi(g^{-1}g_1, c)$ iff $g\psi(x,c) \in p$.

Hence $\mathrm{Stab}_\phi(r) = \mathrm{Stab}_\psi(p)$. As $\bar{p}$ is f-generic and definable over M_0, its stabilizer has bounded index, hence $\mathrm{Stab}_\psi(\bar{p})$ which is an M_0-definable subgroup, has finite index. Completing the proof of (ii).

For the rest, we prove (iii) and (iv) simultaneously. We have seen have that any minimal (closed $G(M_0)$-invariant) subflow $\mathcal{M}$ of $S_G(M_0)$ has the form $\overline{G(M_0)p}$ for $p \in S_G(M_0)$ such that the global heir $\bar{p} \in S(M')$ of p has stabilizer equal to $G^0(M')$. Fix such $\mathcal{M}$ and p. Let a realize $\bar{p}$ (in a bigger model). Then $\mathcal{M} = \{\mathrm{tp}(ga/M_0) : g \in G(M')\}$. As $\mathrm{Stab}(\bar{p}) = G^0(M')$ it is easy to see that the elements of $\mathcal{M}$ are in natural $1-1$ correspondence with the cosets of G^0 in G. This suffices. □

5.7. G **is dp-minimal, proof of Theorem 1.4(4).**

Proposition 5.7. *(Assume NIP.) Let G be a definably amenable, dp-minimal group definable over M_0, and assume all types over M_0 are definable. Then either G has* fsg *(witnessed as usual over M_0), or it has a definable global f-generic type, definable over M_0.*

Proof. Firstly, the existence of a global G-invariant Keisler measure yields trivially a $G(M_0)$-invariant Keisler measure μ over M_0 (i.e. on M_0-definable subsets of G). By Theorem 2.7, μ is definable. So μ extends to a global G-invariant Keisler measure μ' which is definable over M_0. Let p' be a global type in the support of μ'. So p' is $\mathrm{Aut}(\mathbb{M}/M_0)$-invariant. It is proved in [Sim12] that a dp-minimal global type invariant over M_0 is either definable over M_0, or finitely satisfiable in M_0. Now any global type p' in the support of μ' is f-generic and M_0-invariant. If some such type p' is definable over M_0 then we have our global f-generic, definable over M_0. Otherwise all global p' in the support of μ are finitely satisfiable in M_0, whereby G is fsg (with respect to M_0). □

So we can derive part (4) of Theorem 1.4:

Corollary 5.8. *Let G be a definably amenable, dp-minimal group and M any model over which G is defined. Then G/G^{00} coincides with the Ellis group (computed over M^{ext}).*

Proof. It is obvious from the definition of dp-minimality and the fact that $Th(M^{\mathrm{ext}})$ has quantifier elimination, that G remains dp-minimal in $Th(M^{\mathrm{ext}})$, so we can apply Proposition 5.7 together with the parts (1) and (2) of Theorem 1.4 which have already been proved. □

Problem 5.9. Is it true that every dp-minimal group is definably amenable? More specifically, is it true that every dp-minimal group is nilpotent-by-finite?

Remark 5.10. We would expect the Corollary to be true of definably amenable groups of finite dp-rank, by for example finding a composition series of G where the factors are fsg or have definable f-generics. We will see in our proof of part (4) of Theorem 1.4 that this strategy works in the o-minimal case.

On the other hand it is not the case that a definably amenable group of finite dp-rank contains a definable subgroup (or definable quotient) which is dp-minimal. For example, take a real closed field R and take G to be the group of G points of a simple abelian variety over R of algebraic-geometric dimension > 1. Then G has finite dp-rank, has o-minimal dimension > 1 so could not be dp-minimal, and has no proper definable subgroups.

5.8. **The o-minimal case, proof of Theorem 1.4(5).** The remainder of the paper is devoted to proving part (4) of 1.4, the o-minimal case. So we let T be an o-minimal expansion of a real closed field, $M \models T$ and G a (definably connected) definably amenable group, defined over M. We make heavy use of the structure theorem for G given in Section 2 of [CP12]: there is a definable (over M) normal subgroup H of G such that
(i) H is definably connected (solvable) and "torsion-free"
(ii) G/H is definably compact, so fsg by [CP12]. We will denote G/H by T (hopefully without ambiguity) even though T might be noncommutative.

By Proposition 4.7 of [CP12], there is a global, left H-invariant type of H, definable over M.

We now let $M_0 = M^{\mathrm{ext}}$ and pass to $Th(M_0)$. By Theorem 3.19, $G/H = T$ remains fsg and there is still a global H invariant type of H, definable over M_0. In particular H is definably extremely amenable, so $S_H(M_0)$ has a fixed point under the action of $H(M_0)$, hence the unique minimal definable $H(M_0)$-flow is trivial, and *every* definable action of $H(M_0)$ on a compact space has a fixed point. Now the surjective homomorphism $\pi : G \to T$ induces a surjective continuous function $\pi : S_G(M_0) \to S_T(M_0)$, which is clearly also a semigroup homomorphism. Let $\mathcal{M}(G)$ be some minimal subflow of $S_G(M_0)$. So $\pi(\mathcal{M}(G)) = \mathcal{M}(T)$, the unique minimal subflow of $S_T(M_0)$ (which is the set of generic types by Section 5.5). The main point is:

Lemma 5.11. *The restriction of π to $\mathcal{M}(G)$ is a homeomorphism with $\mathcal{M}(T)$.*

Proof. This is a rather general topological dynamics fact (under the hypotheses), surely with a reference somewhere, but we give a proof nevertheless.

The action of $G(M_0)$ on $\mathcal{M}(G)$, induces an action (definable) of $H(M_0)$ on $\mathcal{M}(G)$ which as remarked above (definable extreme amenability of $H(M_0)$) has a fixed point, which we call p. So note that $\mathcal{M}(G) = \overline{G(M_0)p} = \{q \cdot p : q \in S_G(M_0)\}$. As $H(M_0)$ fixes p it follows by definability of p that $H(M')$ fixes $\bar{p}$ where M' is a saturated model, and $\bar{p}$ the unique heir of p over M'. So if $g \in G(M')$ then $g\bar{p}$ depends only on the coset gH. So for $c \in T(M')$, $c\bar{p}$ is well-defined (as $g\bar{p}$ for some/any $g \in G(M')$ such that $gH = c$). Hence we can define $q \cdot p$ for $q \in S_T(M_0)$ as $c\bar{p}|M_0$, namely $\mathrm{tp}(ca/M_0)$ where $c \in T(M')$ realizes q and a realizes $\bar{p}$, and we can easily check that
(i) $\mathcal{M}(G) = \{q \cdot p : q \in S_T(M_0)\}$,
(ii) $T(M_0)$ acts on $\mathcal{M}(G)$, by $c(q \cdot p) = cq \cdot p$.
(iii) Under this action $\mathcal{M}(G)$ is a minimal definable $T(M_0)$-flow.
(iv) $\pi|\mathcal{M}(G)$ is a map of $T(M_0)$-flows.

It follows from (iii) and (iv) that $\pi|\mathcal{M}(G)$ is a homeomorphism with $\mathcal{M}(T)$, as $\mathcal{M}(T)$ is the universal minimal definable $T(M_0)$-flow. (By universality we also have a $T(M_0)$-flow map, $f : \mathcal{M}(T) \to \mathcal{M}(T)$. The composition of f with π has to be an automorphism of $\mathcal{M}(T)$, using Claim (iv) preceding Proposition 3.12 of [GPP12b]). So the lemma is proved. $\square$

By the Lemma and the fact that π is a semigroup homomorphism, π induces and isomorphism between Ellis groups $u\mathcal{M}(G)$ and $\pi(u)\mathcal{M}(T)$. The canonical map $\pi(u)\mathcal{M}(T) \to T/T^{00}$ is an isomorphism by part (1) of Theorem 1.4. On the other hand as $H = H^{00}$ (by Proposition 3.7), it follows that $H < G^{00}$ and hence the map $G \to T$ induces an isomorphism of G/G^{00} with T/T^{00}. So clearly the canonical homomorphism from $u\mathcal{M}(G)$ to G/G^{00} is an isomorphism, as required.

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